

## Aberystwyth University

### *Maximal $L^p$ -regularity for the Laplacian on Lipschitz domains*

Wood, Ian

*Published in:*  
Mathematische Zeitschrift

*DOI:*  
[10.1007/s00209-006-0055-6](https://doi.org/10.1007/s00209-006-0055-6)

*Publication date:*  
2007

*Citation for published version (APA):*  
Wood, I. (2007). Maximal  $L^p$ -regularity for the Laplacian on Lipschitz domains. *Mathematische Zeitschrift*, 255(4), 855-875. <https://doi.org/10.1007/s00209-006-0055-6>

#### **General rights**

Copyright and moral rights for the publications made accessible in the Aberystwyth Research Portal (the Institutional Repository) are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Aberystwyth Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Aberystwyth Research Portal

#### **Take down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

tel: +44 1970 62 2400  
email: [is@aber.ac.uk](mailto:is@aber.ac.uk)

# MAXIMAL $L^p$ -REGULARITY FOR THE LAPLACIAN ON LIPSCHITZ DOMAINS

IAN WOOD

ABSTRACT. We consider the Laplacian with Dirichlet or Neumann boundary conditions on bounded Lipschitz domains  $\Omega$ , both with the following two domains of definition:  $D_1(\Delta) = \{u \in W^{1,p}(\Omega) : \Delta u \in L^p(\Omega), Bu = 0\}$ , or  $D_2(\Delta) = \{u \in W^{2,p}(\Omega) : Bu = 0\}$ , where  $B$  is the boundary operator. We prove that, under certain restrictions on the range of  $p$ , these operators generate positive analytic contraction semigroups on  $L^p(\Omega)$  which implies maximal regularity for the corresponding Cauchy problems. In particular, if  $\Omega$  is bounded and convex and  $1 < p \leq 2$ , the Laplacian with domain  $D_2(\Delta)$  has the maximal regularity property, as in the case of smooth domains. In the last part, we construct an example that proves that, in general, the Dirichlet-Laplacian with domain  $D_1(\Delta)$  is not even a closed operator.

## 1. INTRODUCTION

In this paper we will investigate solutions to the equations

$$(1.1) \quad \left\{ \begin{array}{ll} u' - \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(0) = 0 & \text{in } \Omega, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} u' - \Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega, \\ u(0) = 0 & \text{in } \Omega, \end{array} \right.$$

where  $\Omega \subseteq \mathbb{R}^n$  is a bounded Lipschitz domain, i.e. there exists a constant  $M > 0$  so that every point on the boundary of  $\Omega$  has a neighbourhood  $U$  such that, after an affine change of coordinates,  $\partial\Omega \cap U$  is described by the equation  $x_n = \varphi(x_1, \dots, x_{n-1})$  where  $\varphi$  is a Lipschitz continuous function with Lipschitz constant bounded by  $M$  and  $\Omega \cap U = \{x \in U : x_n > \varphi(x_1, \dots, x_{n-1})\}$ .

Our aim is to prove maximal regularity for the above mentioned problems in the Lebesgue spaces  $L^p(\Omega)$ . To be more precise, what we want to show is that for  $1 < p, q < \infty$  and for every  $f \in L^q(\mathbb{R}_+, L^p(\Omega))$  there exists a unique solution to (1.1) such that  $u' \in L^q(\mathbb{R}_+, L^p(\Omega))$ .

For bounded smooth domains  $\Omega$ , it is well-known that the Laplacian with domain  $D(\Delta) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  generates an analytic  $C_0$ -semigroup on  $L^p(\Omega)$ ,  $1 < p < \infty$  and that it has the maximal regularity property; similarly, in the bounded smooth case, the natural domain for the Neumann-Laplacian lies in  $W^{2,p}(\Omega)$  (see e.g. [DHP03]).

However, in Lipschitz domains this is no longer the case. In [Dah79], Dahlberg constructs a bounded Lipschitz domain  $\Omega \subseteq \mathbb{R}^2$  and a function  $f \in L^\infty(\Omega)$  where the solution to

$$(1.2) \quad \left\{ \begin{array}{ll} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{array} \right.$$

---

The main results of this paper are taken from the author's Ph.D.-thesis, written at the TU Darmstadt under the supervision of Prof. M. Hieber. The author wishes to thank Prof. Hieber for his guidance, encouragement and support in the last few years. Many thanks also go to Prof. C. E. Kenig for his hospitality and many fruitful discussions on the subject during a one-year stay at the University of Chicago.

is not in  $W^{2,p}(\Omega)$  for any  $1 < p < \infty$ . Therefore, in Lipschitz domains, we can no longer expect the maximal regularity results known for smooth domains to hold.

The study of elliptic problems in non-smooth domains was initiated in the 1960's with works by Agmon and Nirenberg [AN63]. The main idea was developed by Kondratiev [Kon67] and has since been extended, in particular by Grubb [Gru86], Maz'ja, Nazarov and Plamenevskij [MP84], [NP94] and Schulze (e.g. [Sch91], [SS99], [KS03]). In this approach, the first step is to consider a model problem in a cone which can be solved by reducing the dimension of the problem using partial Fourier transforms. More general domains can then be treated using localisation methods. Problems in cones and wedges can be treated similarly as done in the books by Kondratiev and Kozlov [KMR01] and Nazarov and Plamenevskij [NP94] and in the parabolic case there are recent results by Solonnikov [Sol01], Nazarov [Naz01] and Prüss and Simonett [PS05] leading to results in weighted  $L^p$ -spaces.

A different approach to the elliptic problem is to use potential theory and harmonic analysis methods to estimate the Green function or harmonic measure. This approach was used by Dahlberg [Dah77], [Dah79], and Jerison and Kenig [JK82], [JK95] for the Dirichlet-Laplacian in bounded Lipschitz domains. It allows to prove uniqueness of the solution to problem (1.2) in certain Sobolev spaces when the data  $f$  is in a negative Sobolev space. Adolfsen [Ado92], [Ado93] and Fromm [Fro93] show that in bounded convex domains  $\Omega$ , or, more generally, bounded Lipschitz domains satisfying a uniform outer ball condition, the solution still gains two degrees of regularity when the data is in  $L^p(\Omega)$ . For the Neumann-Laplacian the corresponding results were proved by Fabes, Mendez and Mitrea [FMM98] and, in the convex case, by Adolfsen and Jerison [AJ94]. In [She95], Shen investigated elliptic systems with constant coefficients in bounded Lipschitz domains. He was able to prove resolvent and gradient estimates in  $L^p(\Omega)$  for  $p$  in an interval around 2. These guarantee that the operator generates a bounded analytic semigroup and that the domain of the operator is contained in  $W^{1,p}(\Omega)$ .

Following Dahlberg's approach to the elliptic problem, Fabes and Salsa [FS83] use caloric measure to solve

$$(1.3) \quad \begin{cases} u_t - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u(0) = 0 & \text{in } \Omega, \\ u = f & \text{on } (0, T) \times \partial\Omega \end{cases}$$

in Lipschitz domains  $\Omega$  for data  $f$  in some  $L^p$ -space on the boundary,  $p \geq 2$ . In [Bro89] and [Bro90], Brown uses potential theoretical arguments to show existence and uniqueness of solutions of (1.3) in certain function spaces and to gain estimates on the non-tangential maximal function of the gradient of the solution. However, none of these methods seem to allow to prove maximal regularity for the operators or to characterise the domain in the  $L^p$ -setting for general Lipschitz domains.

For the parabolic problem

$$\begin{cases} u_t - Au = f & \text{in } (0, T) \times \Omega, \\ B_j u = 0 & \text{on } (0, T) \times \partial\Omega, \quad j = 1, \dots, m \end{cases}$$

in smooth domains  $\Omega$  with an elliptic operator  $A$  of order  $2m$  and boundary operators  $B_j$  there have been many developments in recent years. They rely in large part on developments in operator theory: the analysis of Banach-space valued functions, the functional calculus of sectorial operators and the property of maximal regularity for the linear problem. In [DHP03], maximal  $L^p$ -regularity of solutions to boundary value problems of Agmon-Douglis-Nirenberg type was proved for  $C^{2m}$ -domains.

The aim of this paper is to investigate whether these maximal regularity results extend to the heat equation in Lipschitz domains. To do this, we combine the known results for elliptic equations with results from operator theory to solve the parabolic problem and obtain maximal regularity for the solution. Section 2 introduces the elliptic results we will require later on. The third section then looks at the Dirichlet problem and proves that under certain conditions on the domain and the exponent  $p$ , the Dirichlet-Laplacian generates a  $C_0$ -semigroup in  $L^p(\Omega)$ . Important properties of the semigroup are collected in Section 4. Section 5 repeats the procedure and shows similar results for Neumann boundary conditions, while Section 6 contains the main maximal regularity results which are now an easy consequence of known results from operator theory. Finally, in Section 7 we construct an example proving that restrictions given in Section 6 on the range of exponents  $p$  are necessary for our maximal regularity results.

## 2. KNOWN RESULTS

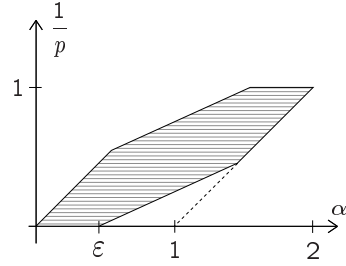
We first state the main result due to Jerison and Kenig on solutions to the Dirichlet problem in bounded Lipschitz domains. The spaces  $L_\alpha^p(\Omega)$  denote the Bessel potential spaces of order  $\alpha$  and exponent  $p$  over the domain  $\Omega$  (cf. [JK95] for the definition and some properties). We remark that for Lipschitz domains  $\Omega$  and  $k \in \mathbb{N}$ , we have  $L_k^p(\Omega) = W^{k,p}(\Omega)$ .

**Theorem 2.1. (Jerison, Kenig [JK95, Theorem 1.1]).** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $n \geq 3$ . There exists  $\varepsilon \in (0, 1]$ , depending only on the Lipschitz character<sup>1</sup> of  $\Omega$  such that for every  $f \in L_{\alpha-2}^p(\mathbb{R}^n)$  there is a unique solution  $u \in L_\alpha^p(\Omega)$  to the inhomogeneous Dirichlet problem*

$$(2.1) \quad \begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

*provided one of the following holds:*

1.  $1 < p \leq p_0$  and  $\frac{3}{p} - 1 - \varepsilon < \alpha < 1 + \frac{1}{p}$ ,
2.  $p_0 < p < p'_0$  and  $\frac{1}{p} < \alpha < 1 + \frac{1}{p}$ ,
3.  $p'_0 \leq p < \infty$  and  $\frac{1}{p} < \alpha < \frac{3}{p} + \varepsilon$ ,



where  $1/p_0 = 1/2 + \varepsilon/2$  and  $1/p'_0 = 1/2 - \varepsilon/2$ . Moreover, we have the estimate

$$\|u\|_{L_\alpha^p(\Omega)} \leq C \|f\|_{L_{\alpha-2}^p(\mathbb{R}^n)}$$

for all  $f \in L_{\alpha-2}^p(\mathbb{R}^n)$ . The constant  $C$  depends on the domain  $\Omega$  only via the Lipschitz character of  $\Omega$ . When the domain is  $C^1$ , the exponent  $p_0$  may be taken to be 1.

**Remark 2.2.** In two dimensions there is a similar result (cf. [JK95, Theorem 1.3]).

For convex bounded domains  $\Omega \subseteq \mathbb{R}^n$  and  $1 < p \leq 2$ , it is possible to control all second derivatives in  $L^p(\Omega)$  by the Laplacian. Note that any bounded convex domain is a Lipschitz domain (cf. [Gri85, Corollary 1.2.2.3]). The following result is due to Fromm.

**Theorem 2.3** (Fromm [Fro93]). *If  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , is a bounded and convex domain with diameter  $d$  and if  $f \in L_\alpha^p(\Omega)$  then there is a unique  $u \in W_0^{1,p}(\Omega) \cap L_{\alpha+2}^p(\Omega)$  satisfying  $\Delta u = f$  in*

<sup>1</sup>That is the number of coordinate charts used to cover the boundary  $\partial\Omega$  by cylinders such that, inside each cylinder, the domain is the domain above the graph of a Lipschitz function, the radii of these cylinders and the supremum of the norms of the graph functions.

$\Omega$ , and this solution satisfies the estimate

$$(2.2) \quad \|u\|_{\alpha+2,p} \leq C(d) \|f\|_{\alpha,p}$$

for  $-1 \leq \alpha \leq 0$  and  $1 < p < \frac{2}{\alpha+1}$  (defining  $\frac{2}{0} = \infty$ ) and for  $\alpha = 0$ ,  $p = 2$ .

The theorem actually holds in a larger class of domains which are Lipschitz domains that are convex in the neighbourhood of any boundary singularities.

**Definition 2.4.** Let  $\Omega$  be a domain in  $\mathbb{R}^n$ . We say that  $\Omega$  satisfies the outer ball condition if for each  $x \in \partial\Omega$ , there exists an open ball  $B \subseteq \Omega^c$  with  $x \in \partial B$ .  $\Omega$  satisfies a uniform outer ball condition if there exists an  $R > 0$  such that for all  $x \in \partial\Omega$ , the ball can be chosen to have radius  $R$ .

**Remark 2.5.** Theorem 2.3 holds in all bounded Lipschitz domains satisfying a uniform outer ball condition (cf. [Fro93, Remarks]). In this case, the constant  $C$  in (2.2) depends on more geometric properties of  $\Omega$  than just the diameter.

The results corresponding to these theorems for Neumann boundary conditions are due to Fabes, Mendez and Mitrea [FMM98] and, in the convex case, Adolfsson and Jerison [AJ94]. They read as follows.

**Theorem 2.6** (Fabes, Mendez, Mitrea [FMM98]). Suppose  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $n \geq 3$ . Then there exists  $\varepsilon \in (0, 1]$ , such that for every  $f \in L^p_{\alpha-2,0}(\Omega)$  satisfying the compatibility condition  $\langle f, \chi_\Omega \rangle = 0$  there is a solution  $u \in L^p_\alpha(\Omega)$  to the inhomogeneous Neumann problem

$$(2.3) \quad \begin{cases} \Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega, \end{cases}$$

provided the pair  $(\alpha, p)$  satisfies the same conditions as stated in Theorem 2.1 for the Dirichlet problem. Moreover, the solution is unique up to a constant and we have the estimate

$$\inf_{c \in \mathbb{C}} \|u - c\|_{L^p_\alpha(\Omega)} \leq C \|f\|_{L^p_{\alpha-2}(\mathbb{R}^n)}.$$

**Remark 2.7.** In fact, in [FMM98] inhomogeneous boundary conditions are considered and the corresponding problems are solved for boundary data in suitable Besov spaces both for the Dirichlet and for the Neumann problem.

As in the case of Dirichlet boundary conditions, for bounded convex domains we can also control the second derivatives.

**Theorem 2.8** (Adolfsson, Jerison [AJ94]). Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded convex domain and  $n \geq 3$ . Suppose  $f \in L^p(\Omega)$  with  $\int_\Omega f = 0$  and  $1 < p \leq 2$ . Then there exists a solution  $u$  to the Neumann problem (2.3) in  $W^{2,p}(\Omega)$  and

$$\int_\Omega |\nabla^2 u|^p \leq C \int_\Omega |f|^p,$$

where  $\nabla^2$  denotes the matrix of the second order derivatives and  $C$  only depends on the Lipschitz character of the domain.

**Remark 2.9.** Note that by Theorem 2.6, the solution is unique up to a constant.

3. GENERATION OF A  $C_0$ -SEMGROUP BY THE DIRICHLET-LAPLACIAN

The aim of this section is to show that if we define the Dirichlet-Laplacian suitably on the Lebesgue spaces  $L^p(\Omega)$ , for certain domains  $\Omega$  and a range of exponents  $p$ , the Dirichlet-Laplacian is the generator of a  $C_0$ -semigroup. We then determine various properties of the generated semigroup. To begin, we introduce the Dirichlet-Laplacian with two different domains of definition. The first of the operators to be introduced is the weak Dirichlet-Laplacian.

**Definition 3.1.** We define the weak Dirichlet-Laplacian  $\Delta_{p,w}^D$  on  $L^p(\Omega)$  by

$$\begin{aligned} D(\Delta_{p,w}^D) &= \{u \in W_0^{1,p}(\Omega) : \Delta u \in L^p(\Omega)\}, \\ \Delta_{p,w}^D u &= \Delta u. \end{aligned}$$

Here,  $\Delta u \in L^p(\Omega)$  is to be understood in the sense of distributions.

In order to obtain results on higher regularity of the solution to the Cauchy problem, we introduce the strong Dirichlet-Laplacian.

**Definition 3.2.** The strong Dirichlet-Laplacian  $\Delta_{p,s}^D$  on  $L^p(\Omega)$  is defined by

$$\begin{aligned} D(\Delta_{p,s}^D) &= W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ \Delta_{p,s}^D u &= \Delta u. \end{aligned}$$

Our aim is to use the Lumer-Phillips Theorem (see e.g. [ABHN01, Theorem 3.4.5]) to prove that the Dirichlet-Laplacian generates a  $C_0$ -semigroup of contractions. To apply the theorem, we need to show dissipativity of the operator.

**Lemma 3.3.** Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded Lipschitz domain and  $2 \leq p < \infty$ . Then  $\Delta_{p,w}^D$  is dissipative.

*Proof.* Let  $u \in D(\Delta_{p,w}^D)$  and set  $u^*(x) = |u(x)|^{p-2} \bar{u}(x) \chi_{\{u \neq 0\}}$ . Then  $u^* \in L^{p'}(\Omega)$  and  $\tilde{u}^* := \frac{u^*}{\|u^*\|_{p'}}$  is in the subdifferential of  $u$ . We also have

$$\nabla u^* = \left( |u|^{p-2} \nabla \bar{u} + \frac{p-2}{2} \bar{u} |u|^{p-4} (\bar{u} \nabla u + u \nabla \bar{u}) \right) \chi_{\{u \neq 0\}}.$$

Using Hölder's inequality, it is easy to check that  $\nabla u^* \in L^{p'}(\Omega)$ . Moreover  $\text{tr } u^* = 0$ . Therefore,  $u^* \in W_0^{1,p'}(\Omega)$ . We can now integrate by parts to obtain

$$(3.1) \quad \text{Re} \langle \Delta u, u^* \rangle = -\text{Re} \int_{\Omega} \nabla u \cdot \nabla u^*.$$

A calculation then yields

$$\begin{aligned} \text{Re} \langle \Delta u, u^* \rangle &= -\text{Re} \int_{\Omega} \nabla u \cdot \nabla u^* \\ &= -\text{Re} \int_{\Omega} |u|^{p-2} |\nabla u|^2 \chi_{\{u \neq 0\}} \\ &\quad -\text{Re} \int_{\Omega} \frac{p-2}{2} \bar{u} |u|^{p-4} (\bar{u} (\nabla u)^2 + u |\nabla u|^2) \chi_{\{u \neq 0\}} \\ &= -\int_{\Omega} (|u|^{p-2} |\nabla u|^2 + (p-2) |u|^{p-4} (\text{Re } \bar{u} \nabla u)^2) \chi_{\{u \neq 0\}} \\ &= -\int_{\Omega} |u|^{p-4} (\bar{u} \nabla u \cdot u \nabla \bar{u} + (p-2) (\text{Re } \bar{u} \nabla u)^2) \chi_{\{u \neq 0\}} \end{aligned}$$

$$= - \int_{\Omega} |u|^{p-4} [(p-1)(\operatorname{Re} \bar{u} \nabla u)^2 + (\operatorname{Im} \bar{u} \nabla u)^2] \chi_{\{u \neq 0\}} \leq 0. \quad \square$$

**Corollary 3.4.** *Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded Lipschitz domain and  $2 \leq p < \infty$ . Then  $\Delta_{p,s}^D$  is dissipative.*

Note that for  $p < 2$ , the function  $u^*$  is not in  $W_0^{1,p'}(\Omega)$ , so the straightforward integration by parts is not possible. However, for the strong Dirichlet-Laplacian an approximation procedure yields the desired result:

**Lemma 3.5.** *Let  $\Omega$  be a Lipschitz domain,  $1 < p < \infty$  and  $u^* = |u|^{p-2}\bar{u}$ . Then for  $u \in W^{2,p}(\Omega)$  we have*

$$(3.2) \quad \begin{aligned} \int_{\Omega} \Delta u \, u^* &= -(p-1) \int_{\Omega} |u|^{p-4} |\operatorname{Re}(\bar{u} \nabla u)|^2 \chi_{\{u \neq 0\}} \\ &\quad - \int_{\Omega} |u|^{p-4} |\operatorname{Im}(\bar{u} \nabla u)|^2 \chi_{\{u \neq 0\}} \\ &\quad - i(p-2) \int_{\Omega} |u|^{p-4} \operatorname{Re}(\bar{u} \nabla u) \operatorname{Im}(\bar{u} \nabla u) \chi_{\{u \neq 0\}} \\ &\quad + \int_{\partial\Omega} \bar{u} |u|^{p-2} \frac{\partial u}{\partial N}. \end{aligned}$$

*Proof.* This is a special case of Theorem 3.1 from [MS04] with  $\phi = 1$  and  $A = \Delta$ . Note that the assumptions on the boundary in [MS04] only require that  $C^\infty(\bar{\Omega})$  is dense in  $W^{2,p}(\Omega)$  and that traces are well defined. In particular the results hold for Lipschitz domains.  $\square$

**Corollary 3.6.** *Let  $\Omega \subseteq \mathbb{R}^n$  be a Lipschitz domain and  $1 < p < \infty$ . Then  $\Delta_{p,s}^D$  is dissipative.*

We are now in the position to prove one of the main theorems of this section for the strong Dirichlet-Laplacian.

**Theorem 3.7.** *Let  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 2$ , be a bounded Lipschitz domain satisfying a uniform outer ball condition and  $1 < p \leq 2$ . Then  $\Delta_{p,s}^D$  generates a  $C_0$ -semigroup of contractions on  $L^p(\Omega)$ .*

*Proof.* It remains to verify the range condition of the Lumer-Phillips-Theorem, i.e. that

$$(\lambda - \Delta)D(\Delta_{p,s}^D) = L^p(\Omega) \text{ for some } \lambda > 0$$

is satisfied. However, from Theorem 2.3, we know that under our assumptions,  $0 \in \rho(\Delta_{p,s}^D)$  and as the resolvent set is open, we have  $\lambda \in \rho(\Delta_{p,s}^D)$  for some small  $\lambda > 0$  which proves the theorem.  $\square$

For the weak Dirichlet-Laplacian we obtain the following result:

**Theorem 3.8.** *Let  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded Lipschitz domain. Then there exists  $\delta > 0$  depending only on the Lipschitz character of  $\Omega$  such that the operator  $\Delta_{p,w}^D$  generates a  $C_0$ -semigroup of contractions on  $L^p(\Omega)$  for  $(3+\delta)' < p < 3+\delta$ , where  $(3+\delta)'$  denotes the conjugate exponent to  $3+\delta$ .*

*If  $n = 2$ , the same statement holds for all  $(4+\delta)' < p < 4+\delta$ .*

*If  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , satisfying a uniform outer ball condition, the statement is true for all  $1 < p < \infty$ .*

*Proof.* We start with the case  $n \geq 3$ . Once again, we verify that the range condition of the Lumer-Phillips-Theorem is satisfied. For  $u \in D(\Delta_{p,w}^D)$ , we have  $\Delta u \in L^p(\Omega)$ , so  $\Delta u \in L_{\alpha-2}^p(\Omega)$  for any  $\alpha \leq 2$ . By Theorem 2.1, there exists  $\delta > 0$  such that whenever  $1 < p < 3 + \delta$ , we can find  $\alpha \in [1, 2]$  and a unique  $v \in L_\alpha^p(\Omega)$  such that

$$\begin{cases} \Delta v = \Delta u & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and  $\|v\|_{\alpha,p} \leq C \|\Delta u\|_{\alpha-2,p}$ . Then  $u - v$  is harmonic and by the Maximum Principle we have  $u = v$ . Thus

$$\|u\|_{1,p} \leq \|u\|_{\alpha,p} \leq C \|\Delta u\|_{\alpha-2,p} \leq C \|\Delta u\|_p.$$

Therefore,  $\Delta_{p,w}^D : D(\Delta_{p,w}^D) \rightarrow L^p(\Omega)$  is an injective mapping for our range of exponents  $p$ . A similar argument shows that it is also surjective. Thus under our assumptions we have  $0 \in \rho(\Delta_{p,w}^D)$  for any  $1 < p < 3 + \delta$ . Moreover, by Lemma 3.3,  $\Delta_{p,w}^D$  is dissipative for  $p \geq 2$ . This proves the theorem for  $2 \leq p < 3 + \delta$ .

For the case when  $(3 + \delta)' < p < 2$  we consider the dual operator. Since  $\Delta_{p',w}^D$  is m-dissipative, its dual operator  $\Delta_{p,w}^D$  is m-dissipative in  $L^p(\Omega)$  (cf. [CH<sup>+</sup>87, Proposition 3.10]).

We now claim that  $\Delta_{p,w}^D \subseteq \Delta_{p',w'}^D$ , i.e.  $D(\Delta_{p,w}^D) \subseteq D(\Delta_{p',w'}^D)$  and both operators coincide on  $D(\Delta_{p,w}^D)$ . To see this, let  $v \in D(\Delta_{p,w}^D)$ ,  $u \in D(\Delta_{p',w'}^D)$ ,  $(v_n) \subseteq C_c^\infty(\Omega)$  and  $(u_n) \subseteq C_c^\infty(\Omega)$  such that  $v_n \rightarrow v$  in  $W^{1,p}(\Omega)$  and  $u_n \rightarrow u$  in  $W^{1,p'}(\Omega)$ . Then

$$\begin{aligned} \langle \Delta u, v \rangle &= \lim_{n \rightarrow \infty} \langle \Delta u, v_n \rangle = - \lim_{n \rightarrow \infty} \langle \nabla u, \nabla v_n \rangle \\ &= - \langle \nabla u, \nabla v \rangle = - \lim_{n \rightarrow \infty} \langle \nabla u_n, \nabla v \rangle \\ &= \lim_{n \rightarrow \infty} \langle u_n, \Delta v \rangle = \langle u, \Delta v \rangle, \end{aligned}$$

so  $v \in D(\Delta_{p',w'}^D)$  and  $\Delta_{p',w'}^D v = \Delta v$  as claimed. Therefore  $\Delta_{p,w}^D$  is contained in a dissipative operator and hence is itself dissipative for  $(3 + \delta)' < p \leq 2$ . Moreover, as we have seen above, for these  $p$  the range condition is satisfied. Using the Lumer-Phillips Theorem, this completes the proof for  $n \geq 3$ .

For  $n = 2$ , we merely replace Theorem 2.1 by [JK95, Theorem 1.3], while in the case of domains satisfying a uniform outer ball condition we use Theorem 2.3 and argue in the same way obtaining the larger range of exponents  $p$ .  $\square$

**Remark 3.9.**  $\delta$  is given by  $3\varepsilon/(1 - \varepsilon)$  where  $\varepsilon$  is the constant given in Theorem 2.1. Therefore,  $(3 + \delta)' = 3/(2 + \varepsilon)$ .

**Corollary 3.10.** *If  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 2$ , is a bounded Lipschitz domain satisfying a uniform outer ball condition and  $1 < p \leq 2$ , then we have  $\Delta_{p,w}^D = \Delta_{p,s}^D$ .*

*Proof.* Obviously,  $\Delta_{p,s}^D \subseteq \Delta_{p,w}^D$ . Furthermore,  $0 \in \rho(\Delta_{p,s}^D) \cap \rho(\Delta_{p,w}^D)$ , so  $\Delta_{p,s}^D = \Delta_{p,w}^D$ .  $\square$

**Remark 3.11.** 1. By similar arguments, it can be shown that the Dirichlet Laplacian with domain

$$D(\Delta_{\alpha,p}^D) = \{f \in L_\alpha^p(\Omega) : \text{tr } f = 0, \Delta f \in L^p(\Omega)\}$$

generates a  $C_0$ -semigroup on  $L_\alpha^p(\Omega)$ ,  $\alpha \in [1, 2]$  whenever the pair  $(\alpha, p)$  satisfies one of the following conditions:

- $3/(2 + \varepsilon) < p < 3/(\alpha - \varepsilon)$  if  $1 \leq \alpha < (3 - \varepsilon)/2$



- $3/(2 + \varepsilon) < p < 2/(1 - \varepsilon)$  if  $(3 - \varepsilon)/2 < \alpha < 3/2$
- $3/(2 + \varepsilon) < p < 1/(\alpha - 1)$  if  $3/2 \leq \alpha < 2$

where  $\varepsilon$  is the constant given in Theorem 2.1 and  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $n \geq 3$ .

3. To make the statements more concise, in the following we will often only refer to the semigroup generated by the weak Dirichlet-Laplacian, recalling that whenever the strong Dirichlet-Laplacian is a generator, it coincides with the weak Dirichlet-Laplacian.

#### 4. PROPERTIES OF THE GENERATED SEMIGROUP

We first show that the semigroups generated on  $L^p(\Omega)$  are consistent.

**Proposition 4.1.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ , where either*

- *$n \geq 3$  and  $(3 + \delta)' < p, q < 3 + \delta$ , where  $\delta > 0$  depends only on the Lipschitz character of  $\Omega$ ,*
- *$n = 2$  and  $(4 + \delta)' < p, q < 4 + \delta$ , where  $\delta > 0$  depends only on the Lipschitz character of  $\Omega$  or*
- *$n \geq 2$  and suppose additionally that  $\Omega$  satisfies a uniform outer ball condition and  $1 < p, q < \infty$ .*

*Then the semigroups  $T_p$  generated by  $\Delta_{p,w}^D$  and  $T_q$  generated by  $\Delta_{q,w}^D$  are consistent, i.e. if  $f \in L^p(\Omega) \cap L^q(\Omega)$  then*

$$T_p(t)f = T_q(t)f \text{ for all } t \geq 0.$$

*Proof.* W.l.o.g. assume  $p < q$ . Then, as  $\Omega$  is bounded,  $D(\Delta_{q,w}^D) \subseteq D(\Delta_{p,w}^D)$ . Let  $f \in D(\Delta_{q,w}^D)$ . Then  $T_q(t)f$  is the unique classical solution to

$$(4.1) \quad u' - \Delta u = 0, \quad u(0) = f$$

with  $T_q(t)f \in D(\Delta_{q,w}^D)$  for  $t \geq 0$ . But then  $T_q(t)f \in D(\Delta_{p,w}^D)$ , so it must agree with  $T_p(t)f$ , the unique classical solution to (4.1) for  $f \in D(\Delta_{p,w}^D)$ . Since  $D(\Delta_{q,w}^D)$  is dense in  $L^q(\Omega)$ , we get  $T_p(t)f = T_q(t)f$  for all  $f \in L^q(\Omega)$ .  $\square$

This now gives us some further interesting results.

**Corollary 4.2.** *Let  $\Omega, n$  and  $p$  be as in Proposition 4.1. Then the semigroup generated by  $\Delta_{p,w}^D$  satisfies a Gaussian estimate, i.e. there exist constants  $a \geq 0$ ,  $M, b > 0$  such that*

$$|T(t)f| \leq Me^{at}G(bt)|f| \text{ for } t \geq 0,$$

*where  $G(t)f = k_t * f$  is the Gaussian semigroup and  $k_t$  is the Gaussian kernel. Moreover, the semigroup generated by  $\Delta_{p,w}^D$  is analytic.*

*Proof.* On  $L^2(\Omega)$ ,  $\Delta_{2,w}^D$  is identical to the Dirichlet-Laplacian defined via the form in [AB99, Section 1]. Since the semigroups generated on  $L^p(\Omega)$  are consistent, both for the construction of the generator in [AB99] and for the semigroups generated by  $\Delta_{p,w}^D$ , the semigroups must coincide for all cases. Then by [AB99, Section 1] the semigroup satisfies a Gaussian estimate. By the Stein Interpolation Theorem, it follows that the semigroup on  $L^p(\Omega)$  is analytic.  $\square$

**Corollary 4.3.** *Under the assumptions on  $\Omega, n$  and  $p$  in Proposition 4.1, the spectrum of  $\Delta_{p,w}^D$  is independent of  $p$ , i.e. for all  $p$  such that  $\Delta_{p,w}^D$  generates a  $C_0$ -semigroup in  $L^p(\Omega)$ , we have  $\sigma(\Delta_{p,w}^D) = \sigma(\Delta_{2,w}^D)$ .*

*Proof.* This follows from the Gaussian estimate by a result due to Kunstmann [Kun99].  $\square$

Our next aim is to show that the generated  $C_0$ -semigroup is positive.

**Lemma 4.4.** *Let  $\Omega, n$  and  $p$  be as in Proposition 4.1. Then the semigroup generated by  $\Delta_{p,w}^D$  is positive in  $L^p(\Omega)$ .*

*Proof.* We can prove the lemma when dealing only with real-valued functions using [CH<sup>+</sup>87, Corollary 7.15] as, in this case,  $L^p(\Omega)$  is a Banach lattice. It is sufficient to show that  $\Delta_{p,w}^D$  is dispersive. In the case  $p \geq 2$ , integration by parts as in the proof of Lemma 3.3 proves this.

When dealing with complex-valued functions, the positivity of the semigroup on  $L^p(\Omega)$ ,  $p \geq 2$ , obviously follows from the real-valued case and the fact that the operator has real-valued coefficients.

Now let  $p < 2$ . Recall that a semigroup is positive iff the resolvent is positive for sufficiently large  $\lambda > 0$  (see e.g. [CH<sup>+</sup>87, Proposition 7.1]). Let  $f \in L^p(\Omega)$ ,  $f \geq 0$ . Then there exist  $f_n \in L^2(\Omega)$ ,  $f_n \geq 0$  such that  $f_n \rightarrow f$  in  $L^p(\Omega)$ . If  $\Delta_{p,w}^D$  generates a  $C_0$ -semigroup on  $L^p(\Omega)$ , then from the resolvent estimate we obtain that  $R(\lambda, \Delta_{p,w}^D)f_n \rightarrow R(\lambda, \Delta_{p,w}^D)f$  in  $L^p(\Omega)$ . However, positivity of the semigroup on  $L^2(\Omega)$  and consistency of the semigroups imply  $R(\lambda, \Delta_{p,w}^D)f_n \geq 0$  and therefore  $R(\lambda, \Delta_{p,w}^D)f \geq 0$  almost everywhere. This proves positivity also for the case  $p < 2$ .  $\square$

We finish this section with a result on the growth bound of the generated semigroup.

**Corollary 4.5.** *Let  $\Omega, n$  and  $p$  satisfy the same assumptions as in Proposition 4.1. Then the semigroup generated by  $\Delta_{p,w}^D$  is of negative type in  $L^p(\Omega)$ , i.e. the growth bound  $\omega(T)$  of the semigroup satisfies  $\omega(T) < 0$ , moreover, it is independent of  $p$ .*

*Proof.* A result due to Weis (cf. [Wei95], [Wei98] or [ABHN01, Theorem 5.3.6] for different versions of the proof), implies that for generators  $A$  of positive semigroups  $T$  on  $L^p(\Omega)$ , we have  $s(A) = \omega(T)$ . By Corollary 4.3, we already know that  $\sigma(\Delta_{p,w}^D) = \sigma(\Delta_{2,w}^D)$ , in particular equality holds for the spectral bound. It therefore remains to examine the case  $p = 2$ .

Using Poincaré's inequality, for  $u \in D(\Delta_{2,w}^D)$ ,  $u \neq 0$ , we have

$$\langle \Delta u, u \rangle = -\|\nabla u\|_2^2 \leq -C\|u\|_2^2 < 0.$$

Then by [Paz83, Theorem 1.3.9], we get that the numerical range and therefore the spectrum of  $\Delta_{2,w}^D$  lie in the half-plane  $\{z \in \mathbb{C} : \operatorname{Re} z \leq -C\}$ , in particular  $s(\Delta_{2,w}^D) \leq -C$ .  $\square$

## 5. GENERATION OF A $C_0$ -SEMIGROUP BY THE NEUMANN-LAPLACIAN

For Neumann boundary conditions, we also introduce the weak and the strong Neumann-Laplacian.

**Definition 5.1.** *We define the weak Neumann-Laplacian  $\Delta_{p,w}^N$  on  $L^p(\Omega)$  by*

$$\begin{aligned} D(\Delta_{p,w}^N) &= \left\{ u \in W^{1,p}(\Omega) : \exists v \in L^p(\Omega) \ \forall \varphi \in W^{1,p'}(\Omega). \ - \int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega} v \varphi \right\}, \\ \Delta_{p,w}^N u &= v. \end{aligned}$$

*We say that  $\frac{\partial u}{\partial N} = 0$  iff*

$$\int_{\Omega} \Delta u \varphi = - \int_{\Omega} \nabla u \nabla \varphi$$

*for any  $\varphi \in W^{1,p'}(\Omega)$ .*

**Definition 5.2.** The strong Neumann-Laplacian  $\Delta_{p,s}^N$  is defined on  $L^p(\Omega)$  by

$$\begin{aligned} D(\Delta_{p,s}^N) &= \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega \right\}, \\ \Delta_{p,s}^N u &= \Delta u. \end{aligned}$$

We will now proceed in a very similar way as for Dirichlet boundary conditions to prove generator results for the Neumann-Laplacian. As solutions to the Neumann problem in bounded domains are only unique up to a constant, we introduce the Banach space

$$L_0^p(\Omega) := \left\{ f \in L^p(\Omega) : \int_{\Omega} f = 0 \right\}$$

of  $L^p$ -functions with mean zero.

Once again, our aim is to use the Lumer-Phillips Theorem to prove generation of a  $C_0$ -semigroup. Therefore, we need to show dissipativity of the operator.

**Lemma 5.3.** Let  $\Omega \subseteq \mathbb{R}^n$  be a Lipschitz domain. Then the weak Neumann-Laplacian  $\Delta_{p,w}^N$  is dissipative in  $L^p(\Omega)$  for  $2 \leq p < \infty$  and the strong Neumann-Laplacian  $\Delta_{p,s}^N$  is for  $1 < p < \infty$ .

*Proof.* First consider the case  $2 \leq p < \infty$ . Define  $u^*$  as in the proof of Lemma 3.3. Since  $u^* \in W^{1,p'}(\Omega)$ , by definition of the boundary condition, we have

$$\operatorname{Re} \langle \Delta u, u^* \rangle = -\operatorname{Re} \int_{\Omega} \nabla u \cdot \nabla u^*.$$

Then the same calculation as in the proof of Lemma 3.3 yields the result.

In the case of the strong Laplacian and  $1 < p < 2$ , we take the real part in formula (3.2).  $\square$

We can now prove generation of a  $C_0$ -semigroup for the strong Neumann-Laplacian.

**Theorem 5.4.** Let  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded convex domain and  $1 < p \leq 2$ . Then  $\Delta_{p,s}^N$  generates a  $C_0$ -semigroup of contractions on  $L^p(\Omega)$ .

*Proof.* It remains to verify the range condition of the Lumer-Phillips-Theorem. From Theorem 2.8, we know that if we restrict the operator  $\Delta_{p,s}^N$  to the space  $L_0^p(\Omega)$  (again denoting the restriction by  $\Delta_{p,s}^N$ ), we have  $0 \in \rho(\Delta_{p,s}^N)$  and as the resolvent set is open, we have  $\lambda \in \rho(\Delta_{p,s}^N)$  for some small  $\lambda > 0$ , i.e. for each  $f \in L_0^p(\Omega)$ , we can find  $u \in W^{2,p}(\Omega) \cap L_0^p(\Omega)$  satisfying the Neumann boundary conditions such that  $(\lambda - \Delta)u = f$ . Given any  $f \in L^p(\Omega)$ , we can write  $f = (f - \bar{f}) + \bar{f}$  where  $\bar{f} := |\Omega|^{-1} \int_{\Omega} f$ . We can then find  $u_0 \in W^{2,p}(\Omega) \cap L_0^p(\Omega)$  satisfying the Neumann boundary conditions such that  $(\lambda - \Delta)u_0 = f - \bar{f}$ . Then  $u := u_0 + \lambda^{-1} \bar{f} \in D(\Delta_{p,s}^N)$  and  $(\lambda - \Delta)u = f$ .  $\square$

**Remark 5.5.** Note that the solution  $u$  given in the proof above is in fact the only solution to  $(\lambda - \Delta)u = f$ , i.e. the operator is also injective and so  $\lambda \in \rho(\Delta_{p,s}^N)$ .

In the weak case, we obtain the following result:

**Theorem 5.6.** Let  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded Lipschitz domain. Then there exists  $\delta > 0$  depending only on the Lipschitz character of  $\Omega$  such that the operator  $\Delta_{p,w}^N$  generates a  $C_0$ -semigroup of contractions on  $L^p(\Omega)$  for  $(3 + \delta)' < p < 3 + \delta$ , where  $(3 + \delta)'$  denotes the conjugate exponent to  $3 + \delta$ .

*Proof.* Once again, we verify that the range condition of the Lumer-Phillips-Theorem is satisfied. Proceeding as in the proof of Theorem 3.8 and using Theorem 2.6 we know that under our assumptions, for any  $1 < p < 3 + \delta$ , the restriction of the operator  $\Delta_{p,w}^N$  is invertible in  $L_0^p(\Omega)$ . As in the proof of the previous theorem we can then verify the range condition. Moreover, by Lemma 5.3,  $\Delta_{p,w}^N$  is dissipative for  $p \geq 2$ . This proves the theorem for  $2 \leq p < 3 + \delta$ .

For the case when  $(3 + \delta)' < p < 2$  we use that the dual operator  $\Delta_{p',w'}^N$  is m-dissipative in  $L^p(\Omega)$  and that  $\Delta_{p,w}^N \subseteq \Delta_{p',w'}^N$ . The latter follows directly from the definition of the boundary conditions. Therefore  $\Delta_{p,w}^N$  is itself dissipative for  $(3 + \delta)' < p \leq 2$ . This completes the proof.  $\square$

**Corollary 5.7.** *If  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 3$ , is a bounded convex domain and  $1 < p \leq 2$ , we have  $\Delta_{p,w}^N = \Delta_{p,s}^N$ .*

We now collect properties of the generated semigroup. The proofs can be copied from the corresponding results for Dirichlet boundary conditions.

The semigroups generated on  $L^p(\Omega)$  are consistent:

**Proposition 5.8.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , where either*

- $(3 + \delta)' < p, q < 3 + \delta$  or
- $\Omega$  is convex and  $1 < p, q < 3 + \delta$ ,

*for some  $\delta > 0$ . Then the semigroups  $T_p$  generated by  $\Delta_{p,w}^N$  and  $T_q$  generated by  $\Delta_{q,w}^N$  are consistent, i.e. if  $f \in L^p(\Omega) \cap L^q(\Omega)$  then*

$$T_p(t)f = T_q(t)f \text{ for all } t \geq 0.$$

**Corollary 5.9.** *Let  $\Omega$  and  $p$  be as in Proposition 5.8. Then the semigroup generated by  $\Delta_{p,w}^D$  satisfies a Gaussian estimate and is analytic.*

**Corollary 5.10.** *Let  $\Omega$  and  $p$  be as in Proposition 5.8. Then the spectrum of  $\Delta_{p,w}^N$  is independent of  $p$ , i.e. for all  $p$  such that  $\Delta_{p,w}^N$  generates a  $C_0$ -semigroup in  $L^p(\Omega)$ , we have  $\sigma(\Delta_{p,w}^N) = \sigma(\Delta_{2,w}^N)$ .*

Finally, we have positivity of the generated  $C_0$ -semigroup.

**Lemma 5.11.** *Let  $\Omega$  and  $p$  be as in Proposition 5.8. Then the semigroup generated by  $\Delta_{p,w}^N$  is positive in  $L^p(\Omega)$ .*

## 6. MAXIMAL REGULARITY FOR THE LAPLACIAN IN BOUNDED LIPSCHITZ DOMAINS

In the previous sections we have proven generator results for the Dirichlet- and Neumann-Laplacian and gathered various properties of the semigroups and their generators. We can now exploit these results to show the desired maximal regularity property for the Laplacian. We start with the case of Dirichlet boundary conditions.

### 6.1. Maximal regularity for the Dirichlet-Laplacian.

**Theorem 6.1.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ , where either*

- $n \geq 3$  and  $(3 + \delta)' < p < 3 + \delta$ , where  $\delta > 0$  depends only on the Lipschitz character of  $\Omega$ ,
- $n = 2$  and  $(4 + \delta)' < p < 4 + \delta$ , where  $\delta > 0$  depends only on the Lipschitz character of  $\Omega$ , or
- $n \geq 2$  and suppose additionally that  $\Omega$  satisfies a uniform outer ball condition and  $1 < p < \infty$ .

Then the weak Dirichlet-Laplacian as defined in Definition 3.1 has the maximal regularity property, i.e. for  $1 < q < \infty$  and for every  $f \in L^q(\mathbb{R}_+, L^p(\Omega))$  there exists a unique solution to

$$(6.1) \quad \begin{cases} u'(t) - \Delta u(t) = f(t) & \text{for } t \in \mathbb{R}_+, \\ u(t, x) = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u(0) = 0. \end{cases}$$

The solution  $u$  lies in  $L^q(\mathbb{R}_+, D(\Delta_{p,w}^D)) \cap W^{1,q}(\mathbb{R}_+, L^p(\Omega))$  and satisfies the estimate

$$\|u\|_{L^q(\mathbb{R}_+, L^p(\Omega))} + \|u'\|_{L^q(\mathbb{R}_+, L^p(\Omega))} + \|\Delta u\|_{L^q(\mathbb{R}_+, L^p(\Omega))} \leq C \|f\|_{L^q(\mathbb{R}_+, L^p(\Omega))}.$$

*Proof.* We have shown that the semigroup generated by  $\Delta_{p,w}^D$  is contractive (Theorems 3.7 and 3.8), analytic (Corollary 4.2) and positive (Lemma 4.4) on  $L^p(\Omega)$ . Maximal regularity now follows from a result by Weis ([Wei01, Corollary 4d]). Moreover, as the generated semigroup is of negative type (Corollary 4.5), we get  $u \in L^q(\mathbb{R}_+, L^p(\Omega))$  (cf. [Dor93]).  $\square$

Of course, whenever the weak and the strong Laplacian coincide, this also yields maximal regularity for the strong Laplacian. Because of the importance of the result, in particular the better estimate (6.2), we state it here separately.

**Theorem 6.2.** For  $1 < p \leq 2$ , and for all bounded Lipschitz domains  $\Omega \subseteq \mathbb{R}^n$ , with  $n \geq 2$ , and  $\Omega$  satisfying a uniform outer ball condition, the strong Dirichlet-Laplacian as defined in Definition 3.2 has the maximal regularity property in  $L^p(\Omega)$ , i.e. for  $1 < q < \infty$  and for every  $f \in L^q(\mathbb{R}_+, L^p(\Omega))$  there exists a unique solution to (6.1). The solution  $u$  lies in  $L^q(\mathbb{R}_+, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap W^{1,q}(\mathbb{R}_+, L^p(\Omega))$  and satisfies the estimate

$$(6.2) \quad \|u\|_{L^q(\mathbb{R}_+, W^{2,p}(\Omega))} + \|u'\|_{L^q(\mathbb{R}_+, L^p(\Omega))} \leq C \|f\|_{L^q(\mathbb{R}_+, L^p(\Omega))}.$$

**Remark 6.3.** The same results can be proven for exterior Lipschitz domains and exterior Lipschitz domains satisfying a uniform outer ball condition. To do this, construct solutions to the resolvent problem in the exterior domain by combining the solutions in bounded domains and the whole space  $\mathbb{R}^n$  as is done in [GHHW05, Theorem 2.5] for Ornstein-Uhlenbeck operators. Together with dissipativity of the operator this proves generation of a contractive  $C_0$ -semigroup via the Lumer-Phillips Theorem. Moreover, we see from the construction that the resolvent operators are consistent on the  $L^p$ -spaces, hence the generated semigroups are consistent and they coincide with the positive analytic semigroups gained by the form method in [AB99]. Invoking Weis' Theorem once more, we obtain maximal  $L^q$ -regularity over  $L^p(\Omega)$  for the Dirichlet-Laplacian in exterior Lipschitz domains with the same restrictions on the dimension, the domain and the exponent  $p$  as in the case of bounded Lipschitz domains.

**6.2. Maximal regularity for the Neumann-Laplacian.** Similar results can now also be obtained for the Neumann-Laplacian. The main difference being that 0 is in the spectrum of the operator and so the solution itself will not lie in the space  $L^q(\mathbb{R}_+, D(\Delta_p^N))$  (cf. [Dor93, Theorem 2.1]).

**Theorem 6.4.** Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , where either

- $(3 + \delta)' < p < 3 + \delta$  or
- $\Omega$  is convex and  $1 < p < 3 + \delta$ ,

for some  $\delta > 0$ . Then the weak Neumann-Laplacian as defined in Definition 5.1 has the maximal regularity property, i.e. for  $1 < q < \infty$  and for every  $f \in L^q(\mathbb{R}_+, L^p(\Omega))$  there exists a unique

solution to

$$(6.3) \quad \begin{cases} u'(t) - \Delta u(t) = f(t) & \text{for } t \in \mathbb{R}_+, \\ \frac{\partial u}{\partial N}(t, x) = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u(0) = 0. \end{cases}$$

The solution  $u$  satisfies the estimate

$$\|u'\|_{L^q(\mathbb{R}_+, L^p(\Omega))} + \|\Delta u\|_{L^q(\mathbb{R}_+, L^p(\Omega))} \leq C \|f\|_{L^q(\mathbb{R}_+, L^p(\Omega))}.$$

*Proof.* Maximal regularity follows from contractivity (Theorems 5.4 and 5.6), analyticity (Corollary 5.9) and positivity (Lemma 5.11) of the semigroup generated by  $\Delta_{p,w}^N$  and the result by Weis ([Wei01, Corollary 4d]).  $\square$

In convex domains we can also control the second derivatives.

**Theorem 6.5.** *For  $1 < p \leq 2$ , and for all bounded convex domains  $\Omega \subset \mathbb{R}^n$  with  $n \geq 3$ , the strong Neumann-Laplacian as defined in Definition 5.2 has the maximal regularity property in  $L^p(\Omega)$ , i.e. for  $1 < q < \infty$  and for every  $f \in L^q(\mathbb{R}_+, L^p(\Omega))$  there exists a unique solution to (6.3). The solution  $u$  satisfies the estimate*

$$\|u'\|_{L^q(\mathbb{R}_+, L^p(\Omega))} + \|\nabla^2 u\|_{L^q(\mathbb{R}_+, L^p(\Omega))} \leq C \|f\|_{L^q(\mathbb{R}_+, L^p(\Omega))}.$$

## 7. NEGATIVE RESULTS

In this final section, we will construct an example that proves that for the weak Dirichlet-Laplacian the upper bound on the exponent  $p$  given in Theorem 6.1 is optimal in bounded Lipschitz domains. This is meant in the sense that for any  $p > 3$ , we can find a bounded Lipschitz domain  $\Omega$  such that the operator  $\Delta_{p,w}^D$  is not even closed in  $L^p(\Omega)$  and therefore it is not the generator of a  $C_0$ -semigroup. Note that it follows easily from Theorem 2.1 that, given a Lipschitz domain  $\Omega$ , there exists  $\varepsilon > 0$  such that  $\Delta_{p,w}^D$  is a closed operator in  $L^p(\Omega)$  for  $1 < p < 3 + \varepsilon$ .

In order to construct the example, we first make some observations on harmonic functions in the complement of a slender cone. Of particular interest will be the behaviour near the tip of the cone. The results that we will need can be found in [KMR01, Sections 2.2 and 2.5.1] and are stated in the next theorem.

**Theorem 7.1.** *Let  $\Gamma_\varepsilon = \{x \in \mathbb{R}^3 : (x_1^2 + x_2^2)^{\frac{1}{2}} \leq -\varepsilon x_3\}$ ,  $\varepsilon > 0$ , be a cone in  $\mathbb{R}^3$ .*

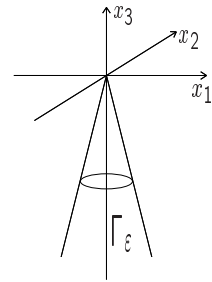
*Consider the Dirichlet problem*

$$(7.1) \quad \begin{cases} \Delta u = 0 & \text{in } \Gamma_\varepsilon^c, \\ u = 0 & \text{on } \partial\Gamma_\varepsilon. \end{cases}$$

*Then there is a solution to (7.1) of the form*

$$u(r, \omega) = r^\lambda \phi(\omega),$$

*where  $r > 0$ ,  $\lambda > 0$ ,  $\omega \in \mathbb{S}^2$  and  $\phi \neq 0$  is a smooth function on the sphere. Furthermore,  $\lambda \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $u$  is positive in  $\Gamma_\varepsilon^c$ .*



Now consider the function  $v$  defined by  $v(r, \omega) := u(r, \omega)\Theta(r)$  for  $r > 0$  and  $\omega \in \mathbb{S}^2$  where  $\Theta$  is a smooth cut-off function supported in  $[0, R]$ . Let  $\Omega = \Gamma_\varepsilon^c \cap B_R$ . Then  $v \in L^p(\Omega)$  for all  $1 < p < \infty$  and has zero boundary values. Moreover,  $f := \Delta v = 2\nabla u \nabla \Theta + u \Delta \Theta$  is in  $L^p(\Omega)$  for all  $1 < p < \infty$  as the gradient of  $\Theta$  vanishes near the tip of the cone  $\Gamma_\varepsilon$ .

By homogeneity we have that for any  $\alpha \in \mathbb{R}_+$ ,  $v \in L_\alpha^p(\Omega)$  iff  $r^{\lambda-\alpha} \in L^p(\Omega)$ , i.e.  $(\lambda - \alpha)p > -3$ . As  $\lambda$  can be made arbitrarily small by making the cone narrower, this implies that if  $p > 3/\alpha$ ,

we can find a bounded Lipschitz domain  $\Omega \subseteq \mathbb{R}^3$  such that  $v \notin L_\alpha^p(\Omega)$ . For any  $1 < p < \infty$ , we can find  $\alpha \in (0, 2)$  such that by Theorem 2.1,  $v$  is the unique solution in  $L_\alpha^p(\Omega)$  to the problem

$$(7.2) \quad \begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, no solution to (7.2) in  $W^{2,p}(\Omega)$  exists whenever  $p > 3/2$ .

**Remark 7.2.** Note that this does not already show that the Laplacian is not closed. It merely proves that the strong Dirichlet-Laplacian is not surjective onto  $L^p(\Omega)$  for  $p > 3/2$  and that the weak Dirichlet-Laplacian is not surjective onto  $L^p(\Omega)$  for  $p > 3$ .

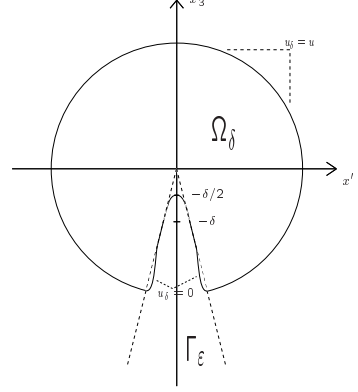
We now use Theorem 7.1 to construct our counterexample. Let  $u(r, \omega) = r^\lambda \phi(\omega)$  be the harmonic function in the complement of the cone  $\Gamma_\varepsilon$  and zero on the boundary given in Theorem 7.1. Fix  $\delta > 0$  and construct  $\Omega_\delta$  by rounding off the tip of the cone at a height between  $\delta/2$  and  $\delta$ , taking the intersection  $\Gamma_\varepsilon^c \cap B_1$  and finally rounding off the corners where the cone intersects the ball as indicated in the figure below. In this way we get a smooth bounded domain  $\Omega_\delta$  containing the set  $\Gamma_\varepsilon^c \cap B_1$  and such that the Lipschitz constant of  $\Omega_\delta$  is uniformly bounded in  $\delta$ .<sup>2</sup>

We solve the following Dirichlet problem.

$$(7.3) \quad \begin{cases} \Delta u_\delta = 0 & \text{in } \Omega_\delta, \\ u_\delta = 0 & \text{on } \partial\Omega_\delta \cap \Gamma_\varepsilon, \\ u_\delta = u & \text{on } \partial\Omega_\delta \cap \Gamma_\varepsilon^c. \end{cases}$$

The solution  $u_\delta$  has the following properties.

- $u_\delta \in C^2(\Omega_\delta) \cap C(\overline{\Omega_\delta})$ : As  $u$  is smooth except for at 0, the boundary data is continuous and since the domain is smooth, we have that  $u_\delta$  is in  $C^2(\Omega_\delta)$  and continuous on the closure of the domain (see e.g. [RR93, Theorem 4.13]). In fact,  $u_\delta \in C^\infty(\Omega_\delta)$  as it is harmonic.
- On  $\Gamma_\varepsilon^c \cap B_1$ , by the Maximum Principle,  $u_\delta \geq u$  and in all of  $\Omega_\delta$ , we have  $u_\delta > 0$ .



Our aim now is to construct a domain and a family of functions  $\{w_\delta\}_{\delta>0}$  such that  $w_\delta$  and  $\Delta w_\delta$  are uniformly bounded in  $L^p(\Omega)$ , but  $\nabla w_\delta$  is not. As a first step, we show a pointwise lower bound for the gradient of  $u_\delta$ .

**Proposition 7.3.** *For  $x$  in a small neighbourhood  $\tilde{\Omega}$  of the origin, we have*

$$|\nabla u_\delta(x)| \geq C \frac{u_\delta(x)}{\text{dist}(x, \partial\Omega_\delta)},$$

where the constant  $C$  depends only on the Lipschitz character of  $\Omega_\delta$ , in particular it is independent of  $\delta$ .

*Proof.* The proof relies on ideas and results taken from [Dah77] and [Caf87]. Using the results of [Caf87], together with [JK82, Lemma 5.4] and the Harnack principle, we can find a small open set  $U$  and constants  $C_1, C_2$  independent of  $\delta$ , such that

$$0 < C_1 \leq \frac{D_{x_3} u_\delta(x', x_3)}{u_\delta(x', x_3)} \leq C_2 \text{ for } x = (x', x_3) \in U.$$

Then a scaling argument yields the desired gradient estimate.  $\square$

<sup>2</sup>In fact, for all  $\delta > 0$ , the Lipschitz constant of  $\Omega_\delta$  can be bounded by the Lipschitz constant of  $\Gamma_\varepsilon$ .

We now need to use a cut-off function to obtain functions which satisfy the zero boundary condition.

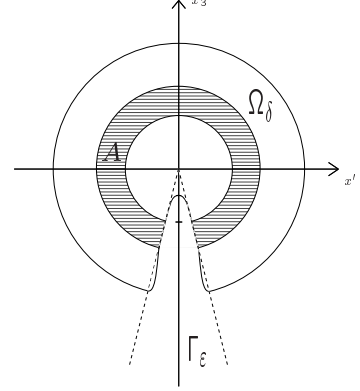
**Proposition 7.4.** *Let  $\Theta \equiv 1$  on  $B_{1/2}$  be a smooth radially symmetric cut-off function supported in  $B_1$ . Let  $w_\delta = \Theta u_\delta$ . Then,*

- a) for  $1 < p < \infty$ ,  $w_\delta \in L^p(\Omega_\delta)$  with uniform bound in  $\delta$ ,
- b) for  $1 < p < \infty$ ,  $\Delta w_\delta \in L^p(\Omega_\delta)$  with uniform bound in  $\delta$ ,
- c) for  $p > 3$ ,  $\left(\int_{\Omega_\delta} |\nabla w_\delta|^p\right)^{\frac{1}{p}} \geq C\delta^{-\zeta}$  for some  $\zeta > 0$ .

*Proof.* a) follows from the Maximum Principle.

b) For the Laplacian of  $w_\delta$  in polar coordinates we have  $\Delta w_\delta = u_\delta \Delta \Theta + 2\partial_r \Theta \partial_r u_\delta$ . The term with  $u_\delta$  is uniformly bounded in  $L^p(\Omega_\delta)$  by the first part of the proof. It remains to show that the term  $\partial_r \Theta \partial_r u_\delta$  is uniformly bounded. To do this we adapt a standard procedure for interior gradient estimates of harmonic functions to our situation. The aim is to construct a subharmonic function involving  $\partial_r u_\delta$  and to apply the Maximum Principle.

We assume w.l.o.g. that the support of  $\nabla \Theta$  is contained within an annulus  $A_0$  such that the intersection of  $A_0$  with the boundary of  $\Omega_\delta$  is contained in the boundary of the cone  $\Gamma_\varepsilon$ . Let  $A = A_0 \cap \Omega_\delta$  (cf. figure). By doing this, we have that, along  $\partial A \cap \partial \Gamma_\varepsilon$ ,  $\partial_r u_\delta$  is the tangential derivative of  $u_\delta$ . Due to the Dirichlet boundary conditions, it then vanishes on this part of  $\partial A$ .



Now set  $v(x) = r \partial_r u_\delta(x)$ .  $v$  is harmonic and  $v^2$  is subharmonic. We could apply the Maximum Principle to  $v$  or  $v^2$ , however we have no knowledge of the behaviour of  $v$  on the boundary of  $A$ . Instead, we introduce another cut-off function  $\zeta \in C_c^\infty(\mathbb{R}^3)$  such that  $\zeta \equiv 1$  on  $\text{supp}(\nabla \Theta)$  and  $\zeta \equiv 0$  on  $A^c$ .

Then, using the first binomial formula,

$$\begin{aligned} \Delta(\zeta^2 v^2) &= (2\zeta \Delta \zeta + 2|\nabla \zeta|^2) v^2 + 8\zeta v \sum_{i=1}^n \partial_i \zeta \partial_i v + 2\zeta^2 |\nabla v|^2 \\ &\geq (2\zeta \Delta \zeta + 2|\nabla \zeta|^2) v^2 - 8|\nabla \zeta|^2 v^2 \geq -Cv^2 \end{aligned}$$

for some  $C \geq 0$  which depends on  $\zeta$  but is independent of  $\delta$ . We can estimate  $v^2 \leq C' |\nabla u_\delta|^2$ . Thus, for some sufficiently large constant  $\alpha$ , the function  $w = \zeta^2 v^2 + \alpha u_\delta^2$  is subharmonic, as

$$\Delta w = \Delta(\zeta^2 v^2) + \Delta(\alpha u_\delta^2) \geq -Cv^2 + 2\alpha |\nabla u_\delta|^2 \geq (-C + 2\alpha/C')v^2 \geq 0.$$

By the Maximum Principle, we have  $\max_A w \leq \max_{\partial A} w$ . However, by construction the part  $\zeta^2 v^2$  vanishes on  $\partial A$  as  $\zeta$  is zero on the two arcs and  $v$  is zero on the radial part of the boundary. Therefore, by part a), we get

$$\max_{\text{supp}(\nabla \Theta)} v^2 \leq \max_A w \leq \max_{\partial A} \alpha u_\delta^2 \leq \alpha M^2,$$

i.e.  $\partial_r u_\delta$  is bounded independently of  $\delta$  on  $\text{supp}(\nabla \Theta)$  which gives a uniform bound on the term  $\partial_r \Theta \partial_r u_\delta$  in  $L^p(\Omega_\delta)$ .



For c), we consider  $w_\delta$  in a shell above the tip of the cone

$$S_\delta = \{(r, \varphi, \theta) : r \in (\delta/2, \delta), \varphi \in [0, 2\pi), \theta \in [0, \alpha)\}$$

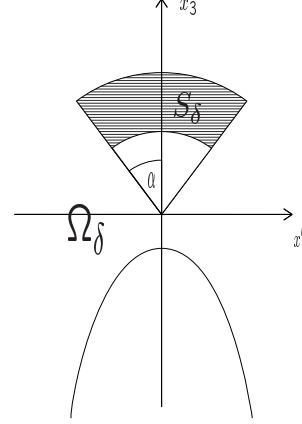
for some small  $\alpha$ , shown in the figure. Assume  $\delta$  is sufficiently small so that  $\Theta \equiv 1$  in  $S_\delta$  and such that the conclusion of Proposition 7.3 holds for all  $x \in S_\delta$ . Then for  $x \in S_\delta$  we have

$$|\nabla w_\delta(x)| = |\nabla u_\delta(x)| \geq C \frac{u_\delta(x)}{\delta} \geq \frac{C}{\delta} u(x) = \frac{C}{\delta} r^\lambda \phi(\omega).$$

Note that, by the maximum principle,  $u > 0$  in  $S_\delta$ , so  $\phi > 0$  and by compactness,  $\phi > c > 0$  in  $S_\delta$ . Therefore,

$$\left( \int_{\Omega_\delta} |\nabla w_\delta|^p \right)^{\frac{1}{p}} \geq \frac{C}{\delta} \delta^\lambda (\mu(S_\delta))^{1/p} = C \delta^{\lambda-1+3/p}.$$

So for any  $p > 3$ , we can make  $\lambda$  sufficiently small to get a negative power of  $\delta$ .  $\square$



We now use this construction to prove the following negative results.

**Theorem 7.5.** *Let  $p > 3$  and let  $\Delta_{p,w}^D$  be the weak Dirichlet-Laplacian defined in Definition 3.1. Then there exist*

- a) *an unbounded smooth domain  $\Omega \subset \mathbb{R}^3$  and*
- b) *a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^3$ ,*

*such that  $\Delta_{p,w}^D$  is not closed in  $L^p(\Omega)$ .*

*Proof.* We start with the unbounded case. Let  $\{\delta_j\}_{j \in \mathbb{N}} \subseteq \mathbb{R}_+$  such that  $\delta_j \rightarrow 0$ . Construct the domains  $\Omega_{\delta_j}$  and functions  $w_{\delta_j}$  as before. Now, put infinitely many cones in a row with tips at  $x_j = (x'_j, 0)$  and with sufficient distance between them such that the sets  $\Omega_{\delta_j} + x_j$  are disjoint. Let  $\Omega$  be a smooth domain above the graph of a function  $\varphi$  containing the sets  $\Omega_{\delta_j} + x_j$ , where, in a neighbourhood of  $x'_j$ ,  $\varphi$  describes the lower boundary of  $\Omega_{\delta_j} + x_j$  (see Figure 1). For  $x \in \Omega$

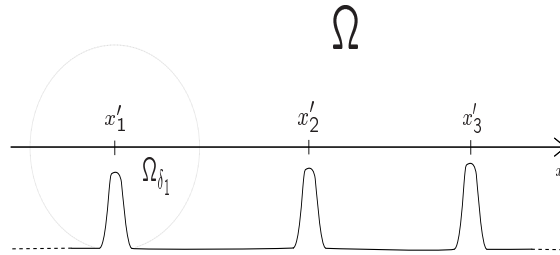


FIGURE 1. The unbounded domain  $\Omega$ .

consider the function

$$w(x) = \sum_{j=0}^{\infty} 2^{-j} w_{\delta_j}(x - x_j),$$

where we extend  $w_{\delta_j}$  to  $\Omega$  by zero outside  $\Omega_{\delta_j}$ . By Proposition 7.4,  $w$  and  $\Delta w$  are in  $L^p(\Omega)$ . We can approximate  $w$  in  $L^p(\Omega)$  by

$$w_N(x) = \sum_{j=0}^N 2^{-j} w_{\delta_j}(x - x_j).$$

Then  $w_N \in D(\Delta_{p,w}^D)$  and  $\Delta w_N(x) \rightarrow \sum_{j=0}^{\infty} 2^{-j} \Delta w_{\delta_j}(x - x_j)$  in  $L^p(\Omega)$ . However, for  $p > 3$ , using that the functions  $w_{\delta_j}$  have disjoint support, we get

$$\left( \int_{\Omega} |\nabla w|^p \right)^{\frac{1}{p}} = \sum_{j=0}^{\infty} 2^{-j} \left( \int_{\Omega_{\delta_j}} |\nabla w_{\delta_j}|^p \right)^{\frac{1}{p}} \geq C \sum_{j=0}^{\infty} 2^{-j} \delta_j^{-\zeta}.$$

Let  $\delta_j \rightarrow 0$  sufficiently fast such that the sum diverges. Then  $w \notin D(\Delta_{p,w}^D)$ , so the operator is not closed.

Next, we consider the case of a bounded domain by using a scaling argument. Let  $\varphi$  be a Lipschitz function describing a cone with the tip at  $(0, 0, 1)$ . Let  $\varphi_{\delta}$  be a smooth function gained from  $\varphi$  by rounding off the tip of the cone between a height of  $1 - \delta$  and  $1 - \delta/2$  and such that  $\|\varphi_{\delta}\|_{\infty} \leq 1$ .

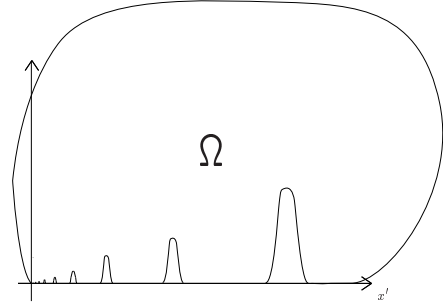
Construct  $w_{\delta}$  as before, allowing for the shift of the tip of the cone away from the origin. For a sequence  $(r_j) \subseteq \mathbb{R}_+$ , define  $w_{\delta,j}(x) = w_{\delta}(r_j^{-1}x)$  for  $x \in \Omega_{\delta,j} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > r_j \varphi_{\delta}(r_j^{-1}x_1, r_j^{-1}x_2)\}$ . The scaled domains all have the same Lipschitz constant.

$w_{\delta,j}$  has the following properties.

$$\begin{aligned} \int_{\Omega_{\delta,j}} |w_{\delta,j}(x)|^p &= r_j^3 \int_{\Omega_{\delta}} |w_{\delta}|^p && \leq C r_j^3, \\ \int_{\Omega_{\delta,j}} |\Delta w_{\delta,j}(x)|^p &= r_j^{3-2p} \int_{\Omega_{\delta}} |\Delta w_{\delta}|^p && \leq C r_j^{3-2p}, \\ \int_{\Omega_{\delta,j}} |\nabla w_{\delta,j}(x)|^p &= r_j^{3-p} \int_{\Omega_{\delta}} |\nabla w_{\delta}|^p && \geq C r_j^{3-p} \delta^{-\zeta p}. \end{aligned}$$

Let  $x_j = (2^{-j}, 0, 0)$ ,  $r_j = 8^{-j}$  and

$$w(x) = \sum_{j=0}^{\infty} 8^{-3j(1-1/p)} w_{\delta,j}(x - x_j).$$



Then  $w$  and  $\Delta w$  are in  $L^p(\Omega)$  where  $\Omega$  is a bounded domain containing the sets  $\Omega_{\delta,j} + x_j$  with its lower boundary given by  $\sum_{j=0}^{\infty} r_j \varphi_{\delta}(r_j^{-1}(x - x_j))^3$  as in the figure. Now let  $\delta_j \rightarrow 0$  sufficiently fast, so that  $\nabla w \notin L^p(\Omega)$  for  $p > 3$ . By approximating  $w$  by the partial sums, we see that  $\Delta_{p,w}^D$  is not closed on  $L^p(\Omega)$ .  $\square$

In higher dimensions, we get the same result.

**Corollary 7.6.** *Let  $p > 3$  and  $n \geq 3$ . Then there exist*

- a) *an unbounded smooth domain  $\Omega \subseteq \mathbb{R}^n$  and*
- b) *a bounded Lipschitz domain  $\Omega \subseteq \mathbb{R}^n$ ,*

*such that  $\Delta_{p,w}^D$  is not closed in  $L^p(\Omega)$ .*

<sup>3</sup>This is necessary, as it is not possible to smoothly continue the functions  $w_{\delta,j}$  by zero across this part of the boundary of  $\Omega_{\delta,j}$ . Also note that  $\Omega$  is Lipschitz but not  $C^1$  as the derivatives of the function describing the boundary do not converge at zero.

*Proof.* We can use the same construction as in  $\mathbb{R}^3$  by setting  $\mathbb{R}^n = \mathbb{R}^3 \times \mathbb{R}^{n-3}$ . Then let  $\tilde{u}(r, \omega, x) = u(r, \omega)$  for  $r \in \mathbb{R}_+$ ,  $\omega \in \mathbb{S}^2 \cap \Gamma_\varepsilon^c$  and  $x \in \mathbb{R}^{n-3}$  with  $\Gamma_\varepsilon$  and  $u$  as in Theorem 7.1. Then we can construct smooth bounded domains  $\Omega_\delta \subseteq \mathbb{R}^n$  and functions  $u_\delta$  and  $w_\delta$  as in the case of three dimensions.  $\square$

## REFERENCES

- [AB99] W. Arendt and P. B enilan. Wiener regularity and heat semigroups on spaces of continuous functions. *Progress in Nonlinear Diff. Eq. and their applications* **35**, 1999. 29-49.
- [ABHN01] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*. Monographs in Mathematics **96**. Birkh auser, Basel, 2001.
- [Ado92] V. Adolfsson.  $L^2$ -integrability of the second order derivatives for Poisson's equation in nonsmooth domains. *Math. Scand.* **70**, 1992. 146-160.
- [Ado93] V. Adolfsson.  $L^p$ -integrability of the second order derivatives of Green potentials in convex domains. *Pac. J. Math.* **159**, 2, 1993. 201-225.
- [AJ94] V. Adolfsson and D. Jerison.  $L^p$ -integrability of the second order derivatives for the Neumann problem in convex domains. *Indiana Univ. Math. J.* **43**, 4, 1994. 1123-1138.
- [AN63] S. Agmon and L. Nirenberg. Properties of solutions of ordinary differential equations in Banach space. *Comm. Pure and Appl. Math.* **16**, 1963. 121-239.
- [Bro89] R. M. Brown. The method of layer potentials for the heat equation in Lipschitz cylinders. *Amer. J. Math.* **111**, 1989. 339-379.
- [Bro90] R. M. Brown. The initial-Neumann Problem for the heat equation in Lipschitz cylinders. *Trans. AMS* **320**, 1990. 1-52.
- [Caf87] L. A. Caffarelli. A Harnack inequality approach to the regularity of free boundaries. Part I: Lipschitz free boundaries are  $C^{1,\alpha}$ . *Rev. Mat. Iberoamericana* **3**, 2, 1987. 139-162.
- [CH<sup>+</sup>87] P. Cl ement, H. J. A. M. Heijmans, et al. *One-Parameter Semigroups*. CWI Monographs **5**. North-Holland, Amsterdam, 1987.
- [Dah77] B. E. J. Dahlberg. On estimates of harmonic measure. *Arch. Rational Mech. Anal.* **65**, 1977. 272-288.
- [Dah79] B. E. J. Dahlberg.  $L^q$ -estimates for Green potentials in Lipschitz domains. *Math. Scand.* **44**, 1979. 149-170.
- [DHP03] R. Denk, M. Hieber, and J. Pr uss.  $\mathcal{R}$ -boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Memoirs Amer. Math. Soc.*, 2003. 1-114.
- [Dor93] G. Dore.  $L^p$ -regularity for abstract differential equations. In: *Functional Analysis and Related Topics*, H. Komatsu (ed.), Lecture Notes in Math. **1540**, Springer, 1993.
- [FMM98] E. Fabes, O. Mendez, and M. Mitrea. Boundary layers on Sobolev-Besov Spaces and Poisson's equation for the Laplacian in Lipschitz domains. *J. Func. Anal.* **159**, 1998. 323-368.
- [Fro93] S. J. Fromm. Potential space estimates for Green potentials in convex domains. *Proc. Amer. Math. Soc.* **119**, 1, 1993. 225-233.
- [FS83] E. Fabes and S. Salsa. Estimates of caloric measure and the initial-Dirichlet problem for the heat equation in Lipschitz cylinders. *Trans. AMS* **279**, 1983. 635-650.
- [GHHW05] M. Ge bert, H. Heck, M. Hieber, and I. Wood. The Ornstein-Uhlenbeck semigroup in exterior domains. *Preprint*, 2005.
- [Gri85] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston, 1985.
- [Gru86] G. Grubb. *Functional Calculus of Pseudodifferential Boundary Problems*. Birkh auser, Boston, 1986.
- [JK82] D. Jerison and C. Kenig. Boundary value problems in Lipschitz domains. In: *Studies in Partial Differential Equations*, W. Littmann (ed.), MAA Studies in Math. **23**, 1982. 1-68.
- [JK95] D. Jerison and C. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Func. Anal.* **130**, 1995. 161-219.
- [KMR01] V. A. Kozlov, V. G. Maz'ya, and J. Ro mann. *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*. Mathematical Surveys and Monographs **85**. AMS, 2001.
- [Kon67] V. A. Kondratiev. Boundary problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.* **16**, 1967. 227-313.
- [KS03] D. Kapanadze and B.-W. Schulze. Symbolic calculus for boundary value problems on manifolds with edges. *Integr. Equ. Oper. Theory* **45**, 1, 2003. 64-104.
- [Kun99] P. C. Kunstmann. Heat kernel estimates and  $L^p$  spectral independence of elliptic operators. *Bull. London Math. Soc.* **31**, 1999. 345-353.

- [MP84] V. G. Maz'ja and B. A. Plamenevskij. Estimates in  $L_p$  and Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary. *Amer. Math. Soc. Transl.* **123**, 1984. 1-56.
- [MS04] G. Metafune and C. Spina. An intergration by parts formula in Sobolev spaces. *Ulmer Seminare über Funktionalanalysis und Differentialgleichungen*, 2004.
- [Naz01] A. I. Nazarov.  $L_p$ -estimates for a solution to the Dirichlet problem and to the Neumann problem for the heat equation in a wedge with edge of arbitrary codimension. *J. Math. Sci.* **106**, 3, 2001. 2989-3014.
- [NP94] S. A. Nazarov and B. A. Plamenevskij. *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. De Gruyter, Berlin, 1994.
- [Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.
- [PS05] J. Prüss and G. Simonett.  $H^\infty$ -calculus for the sum of noncommuting operators. *Preprint*, 2005.
- [RR93] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*. Springer, 1993.
- [Sch91] B.-W. Schulze. *Pseudo-Differential Operators on Manifolds with Singularities*. North Holland, Amsterdam, 1991.
- [She95] Z. Shen. Resolvent estimates in  $L^p$  for elliptic systems in Lipschitz domains. *J. Func. Anal.* **133**, 1995. 224-251.
- [Sol01] V. A. Solonnikov.  $L_p$ -estimates for solutions of the heat equation in a dihedral angle. *J. Math. Sci.* **106**, 3, 2001. 2989-3014.
- [SS99] E. Schrohe and B.-W. Schulze. Mellin and Green symbols for boundary value problems in manifolds with edges. *Integral Equations Oper. Theory* **34**, 3, 1999. 339-363.
- [Wei95] L. Weis. The stability of positive semigroups on  $L_p$  spaces. *Proc. Amer. Math. Soc.* **123**, 1995. 3089-3094.
- [Wei98] L. Weis. A short proof for the stability theorem for positive semigroups on  $L_p(\mu)$ . *Proc. Amer. Math. Soc.* **126**, 1998. 3253-3256.
- [Wei01] L. Weis. A new approach to maximal  $L^p$ -regularity. In: *Evolution Equations and Appl. Physical Life Sciences*, G.Lumer, L.Weis (eds.), *Lect. Notes in Pure & Appl. Math.* **215** Marcel Dekker, New York, 2001. 195-214.

FACHBEREICH MATHEMATIK, ANGEWANDTE ANALYSIS, TECHNISCHE UNIVERSITÄT DARMSTADT, SCHLOSSGARTENSTR. 7, D-64289 DARMSTADT, GERMANY

E-mail address: wood@mathematik.tu-darmstadt.de